A Coupled Oscillator Model for the Acoustic Guitar

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Abstract

The low frequency response of an acoustic guitar is calculated by following the model proposed by Christensen and Vistisen in which the soundhole air column and top plate are treated as coupled oscillators.[1] It is concluded that the lowest two resonances of the guitar are a direct result of the coupling.

Contents

I. INTRODUCTION

As a popular instrument, the acoustic guitar can provide common, real-life demonstrations of acoustic phenomena such as traveling waves, beats and resonances. Here we are particularly interested in using a simple model to predict the resonances of acoustic guitars made of different materials. Such predictions can guide the selection of materials to produce guitars with different acoustic properties to suit various musical tastes. The model described here consists of two coupled oscillators and can reproduce the qualitative features of the response of an acoustic guitar that is forced into oscillations at frequencies less than approximately 1000 Hz.

In Section II I describe the coupled oscillator model, starting with the equations of motion for the individual oscillators. In Section III I compare the calculated sound pressure level, mobility level, and top plate velocity phase to the results of Christensen and Vistisen. I also discuss the utility of the model. Appendix A is a mathematical justification for the substitution of new variables, and Appendix B is a listing of the Mathematica code used to generate the results presented in this paper. Lastly, Appendix C contains information regarding the presentation I gave at the 2007 Eastern Michigan University Graduate Research Fair.

II. THE COUPLED OSCILLATOR MODEL

The vibrations of guitar strings by themselves produce negligible sound (what listeners often refer to as volume). However, the strings are fastened to the guitar body top plate to which they transmit their vibrations. By causing the top plate to vibrate, the strings are said to "drive" the top plate. The vibrating top plate in turn causes the air inside the cavity to vibrate. The air inside the cavity then drives the small column of air located in the soundhole, or the Helmholtz oscillator, of the guitar. Most of the sound produced by the guitar is generated by the vibrations of the top plate and of the soundhole air column, which are "coupled" by the vibrations of the air cavity.

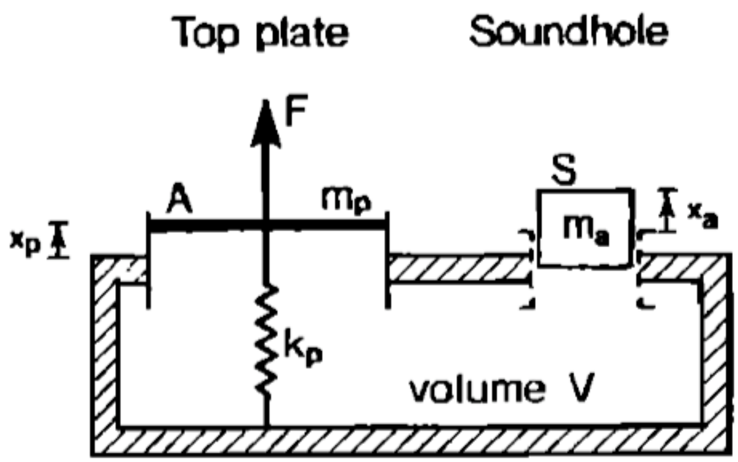

FIG. 1: The top plate is coupled to the air column resident in the soundhole via the air cavity, which functions like a spring. Note that the spring in the diagram is a representation of the stiffness of the top plate.[1]

Typical guitars have six or twelve strings and most guitar players play all strings simultaneously (strumming chords, for example). Determining the response of an acoustic guitar to a single sinusoidal driving frequency may seem like oversimplifying a complex system but as I will demonstrate, it can be done accurately for low frequencies. A guitar can be driven monotonically in several ways. The most typical experimental methods involve attaching a transducer to the top plate. A low-cost method to drive the top plate involves affixing a small loudspeaker with the cardboard cone or membrane removed to the top plate, ensuring a low-mass speaker so as to avoid mass loading. Either excitation source would be controlled by a frequency generator. Measurements of the resulting top plate motion could be made by a simple microphone, accelerometer, or a laser interferometer.

We will follow the model described by Christensen and Vistisen. We begin by obtaining the equations of motion for the top plate and air column. We solve both equations by assuming steady state solutions of a single frequency. We then calculate the sound pressure level and the top plate mobility as a function of the driving frequency. Finally, we calculate the phase between top plate mobility and the driving force as a function of driving frequency.

We start to obtain the equations of motion by describing the change in volume of the air cavity. Taking the outward motion of the oscillators to be positive, the change in volume of the cavity is

$$\Delta V = Ax_p + Sx_a \tag{1}$$

where x_p is the position of the top plate, x_a is the position of the air piston, A is the area of the top plate, and S is the area of the soundhole, respectively.

For adiabatic compression, the change in cavity pressure Δp, is a function of the volume V, change in volume ΔV, the speed of sound in air c, and the density of air ρ:[2]

$$\Delta p = -\mu \Delta V \tag{2}$$

where

$$\mu \equiv c^2 \rho / V \tag{3}$$

The top plate is driven by a transducer at a frequency ν (angular frequency $\omega = 2\pi\nu$) with a force of excitation f. The top plate has a stiffness of k_p and a damping force of magnitude $R_p \dot{x}_p$ opposing motion, where R_p is a characteristic of the material, aptly termed resistance to motion. Likewise, the air piston has a resistance to motion of R_a. Incorporating these variables into the equation of motion for the top plate, we obtain

$$m_p \ddot{x}_p = f - k_p x_p - R_p \dot{x}_p + A\Delta p \tag{4}$$

Inserting our expression for Δp into Eq. (4) yields

$$m_p \ddot{x}_p = f - x_p(k_p + \mu A^2) - R_p \dot{x}_p - \mu S A x_a \tag{5}$$

Solving for $\ddot{x}_p$, we obtain

$$\ddot{x}_p = \frac{f}{m_p} - x_p \frac{(k_p + \mu A^2)}{m_p} - \frac{R_p}{m_p} \dot{x}_p - \frac{\mu S A}{m_p} x_a \tag{6}$$

The equation of motion of the air piston is

$$m_a\ddot{x}_a = S\Delta p - R_a\dot{x}_a \tag{7}$$

Inserting our expression for Δp into Eq.(7) yields

$$m_a\ddot{x}_a = -\mu S^2 x_a - R_a\dot{x}_a - \mu SAx_p \tag{8}$$

Equations (6) and (8) are the equations of motion for the top plate and air piston, respectively. These equations are coupled because they each depend on the coordinates x_p and x_a.

We now solve the equations of motion. Rearranging Equation (8), we obtain

$$m_a\ddot{x}_a + R_a\dot{x}_a + \mu S^2 x_a = -\mu SAx_p \tag{9}$$

Assuming that x_a has a steady state solution of the form

$$x_a = X_a e^{i\omega t} \tag{10}$$

and x_p has a steady state solution of the form

$$x_p = X_p e^{i\omega t} \tag{11}$$

then Eq.(9) becomes

$$m_a(i\omega)^2 X_a e^{i\omega t} + R_a(i\omega)X_a e^{i\omega t} + \mu S^2 X_a e^{i\omega t} = -\mu SAX_p e^{i\omega t} \tag{12}$$

Dividing through by $e^{i\omega t}$ and collecting like terms, we obtain

$$X_a(-m_a\omega^2 + i\omega R_a + \mu S^2) = X_p(-\mu SA) \tag{13}$$

Solving for X_a yields

$$X_a = \frac{\mu SA}{m_a\omega^2 - i\omega R_a - \mu S^2} X_p \tag{14}$$

Now we have an expression for X_a in terms of X_p. Now we will return to Eq.(6) and solve for X_p in terms of X_a. Assuming that the driving force has the form

$$f = Fe^{i\omega t} \tag{15}$$

and inserting Eqs. (10), and Eq. (11) into Eq. (6), we obtain

$$-\omega^2 X_p e^{i\omega t} + \frac{i\omega R_p}{m_p} X_p e^{i\omega t} + \frac{k_p + \mu A^2}{m_p} X_p e^{i\omega t} = \frac{Fe^{i\omega t}}{m_p} - \frac{\mu SA}{m_p} X_a e^{i\omega t} \tag{16}$$

Substituting X_a from Eq. (14) and canceling factors of $e^{i\omega t}$

$$-\omega^2 X_p + \frac{i\omega R_p}{m_p} X_p + \frac{k_p + \mu A^2}{m_p} X_p = \frac{F}{m_p} - \frac{\mu S A}{m_p}\left(\frac{\mu S A}{m_a\omega^2 - i\omega R_a - \mu S^2}\right) X_p \tag{17}$$

Rearranging the last term we have

$$-\omega^2 X_p + \frac{i\omega R_p}{m_p} X_p + \frac{k_p + \mu A^2}{m_p} X_p = \frac{F}{m_p} - \frac{\mu A^2}{m_p}\left(\frac{\mu S^2}{m_a\omega^2 - i\omega R_a - \mu S^2}\right) X_p \tag{18}$$

The plate resonance frequency for a closed cavity (where $S = 0$) is

$$\omega_p \equiv [(k_p + \mu A^2)/m_p]^{1/2} \tag{19}$$

Using Eq. (19), the plate resonance frequency for a closed cavity with a spring constant of zero ($k_p = 0$) would be

$$\omega_a \equiv (\mu A^2/m_p)^{1/2} \tag{20}$$

A zero spring constant indicates zero restoring force; in other words, the mass remains in the position to which it is displaced. Substitution of Eqs. (19) and (20) into Eq.(18) yields

$$-\omega^2 X_p + \frac{i\omega R_p}{m_p} X_p + \omega_p^2 X_p = \frac{F}{m_p} - \omega_a^2\left(\frac{\mu S^2}{m_a\omega^2 - i\omega R_a - \mu S^2}\right) X_p \tag{21}$$

Dividing the numerator and denominator of the last term in this equation by m_a allows us to introduce the Helmholtz resonance frequency ω_h. Any container with an opening smaller than the wavelength of the sound produced has a resonant frequency at which the air resident in the opening not the cavity - oscillates at maximum amplitude. The Helmholtz resonance frequency with the top plate position x_p held at zero by clamps is

$$\omega_h \equiv \left(\frac{\mu S^2}{m_a}\right)^{1/2} \tag{22}$$

We now have

$$-\omega^2 X_p + \frac{i\omega R_p}{m_p} X_p + \omega_p^2 X_p = \frac{F}{m_p} - \frac{\omega_h^2\omega_a^2}{\omega^2 - i\omega\frac{R_a}{m_a} - \omega_h^2} X_p \tag{23}$$

Now we introduce the damping coefficients of the plate and the air piston in an uncoupled system:

$$\gamma_a \equiv R_a/m_a \tag{24}$$

$$\gamma_p \equiv R_p/m_p \tag{25}$$

Inserting γ_a and γ_p into Eq. (23) we have

$$-\omega^2 X_p + i\omega\gamma_p X_p + \omega_p^2 X_p = \frac{F}{m_p} - \frac{\omega_h^2\omega_a^2}{\omega^2 - i\omega\gamma_a - \omega_h^2} X_p \tag{26}$$

Gathering the X_p terms to the left hand side, we obtain

$$X_p \left(-\omega^2 + i\omega\gamma_p + \omega_p^2 + \frac{\omega_h^2\omega_a^2}{\omega^2 - i\omega\gamma_a - \omega_h^2} \right) = \frac{F}{m_p} \tag{27}$$

Next we multiply through by $\omega^2 - i\omega\gamma_a - \omega_h^2$

$$X_p[(-\omega^2 + i\omega\gamma_p + \omega_p^2)(\omega^2 - i\omega\gamma_a - \omega_h^2) + \omega_h^2\omega_a^2] = \frac{F}{m_p}(\omega^2 - i\omega\gamma_a - \omega_h^2) \tag{28}$$

Multiplying both sides by -1 and rearranging terms

$$X_p \left[(\omega_p^2 - \omega^2 + i\omega\gamma_p)(\omega_h^2 - \omega^2 + i\omega\gamma_a) - \omega_h^2\omega_a^2 \right] = \frac{F}{m_p}(\omega_h^2 - \omega^2 + i\omega\gamma_a) \tag{29}$$

Solving for X_p yields

$$X_p = \frac{\frac{F}{m_p}(\omega_h^2 - \omega^2 + i\omega\gamma_a)}{(\omega_p^2 - \omega^2 + i\omega\gamma_p)(\omega_h^2 - \omega^2 + i\omega\gamma_a) - \omega_h^2\omega_a^2} \tag{30}$$

To obtain top plate velocity u_p, we differentiate Eq.(11) with respect to time:

$$\dot{x}_p = u_p = U_p e^{i\omega t} = X_p(i\omega)e^{i\omega t} \tag{31}$$

Therefore

$$U_p = i\omega X_p = i\omega \frac{\frac{F}{m_p}(\omega_h^2 - \omega^2 + i\omega\gamma_a)}{(\omega_p^2 - \omega^2 + i\omega\gamma_p)(\omega_h^2 - \omega^2 + i\omega\gamma_a) - \omega_h^2\omega_a^2} \tag{32}$$

Defining ω_{ph} as our coupling frequency

$$\omega_{ph}^4 \equiv \omega_h^2\omega_a^2 \tag{33}$$

and defining the denominator as

$$D \equiv (\omega_p^2 - \omega^2 + i\omega\gamma_p)(\omega_h^2 - \omega^2 + i\omega\gamma_a) - \omega_{ph}^4 \tag{34}$$

we can now rewrite Eq. (32) as

$$U_p = i\omega(F/m_p)\left[[(\omega_h^2 - \omega^2) + i\omega\gamma_a]/D \right] \tag{35}$$

To determine u_a, the air column velocity, we must return to X_a. Recall that

$$X_a = \frac{\mu S A}{m_a \omega^2 - i\omega R_a - \mu S^2} X_p \tag{36}$$

Dividing the numerator and denominator by m_a

$$X_a = \frac{\frac{\mu S A}{m_a}}{\omega^2 - i\omega \frac{R_a}{m_a} - \frac{\mu S^2}{m_a}} X_p \tag{37}$$

Substituting Eq. (22) and our definition for γ_a into this equation we have

$$X_a = \frac{\frac{A}{S}\omega_h^2}{\omega^2 - i\omega\gamma_a - \omega_h^2} X_p \tag{38}$$

Next we insert Eq. (30) for X_p into Eq. (38)

$$X_a = \frac{\frac{A}{S}\omega_h^2}{\omega^2 - i\omega\gamma_a - \omega_h^2} \times \frac{\frac{F}{m_p}(\omega_h^2 - \omega^2 + i\omega\gamma_a)}{(\omega_p^2 - \omega^2 + i\omega\gamma_p)(\omega_h^2 - \omega^2 + i\omega\gamma_a) - \omega_h^2\omega_a^2} \tag{39}$$

The denominator on the right is equal to D. We also factor out a negative sign from the numerator on the right.

$$X_a = \frac{\frac{A}{S}\omega_h^2}{\omega^2 - \omega_h^2 - i\omega\gamma_a} \times \frac{-\frac{F}{m_p}(\omega^2 - \omega_h^2 - i\omega\gamma_a)}{D} \tag{40}$$

Canceling and collecting appropriate terms we have

$$X_a = -(F/m_p)(A/S)(\omega_h^2/D) \tag{41}$$

Differentiating Eq. (10) yields

$$\dot{x}_a = u_a = U_a e^{i\omega t} = X_a(i\omega)e^{i\omega t} \tag{42}$$

Therefore

$$U_a = i\omega X_a = -i\omega(F/m_p)(A/S)(\omega_h^2/D) \tag{43}$$

When the denominator approaches zero, U_a approaches infinity and the air column is said to be vibrating in resonance. Assuming negligible damping, i.e., $\omega\gamma_p << \omega_p^2 - \omega^2$ and $\omega\gamma_a << \omega_h^2 - \omega^2$, we can set the resulting denominator equal to zero and determine the resonant frequencies of the air piston.

$$(\omega_p^2 - \omega^2)(\omega_h^2 - \omega^2) - \omega_{ph}^4 = 0 \tag{44}$$

This can be solved using the quadratic equation. To make it a little easier to see, I set $\omega_p^2 = a$, $\omega_h^2 = b$, $\omega_{ph}^4 = c$ and $\omega^2 = x$.

$$(a - x)(b - x) - c = 0 \tag{45}$$

$$x^2 - ax - bx + ab - c = 0 \tag{46}$$

$$x^2 + (-a - b)x + (ab - c) = 0 \tag{47}$$

Using the quadratic formula, we find

$$x = \frac{(a + b) \pm \left[(a + b)^2 - 4(ab - c)\right]^{1/2}}{2} \tag{48}$$

Separating $(a + b)$ we have

$$x = \frac{a + b}{2} \pm \frac{\left[(a^2 + 2ab + b^2) - 4ab + 4c\right]^{1/2}}{2} \tag{49}$$

$$x = \frac{a + b}{2} \pm \frac{(a^2 - 2ab + b^2 + 4c)^{1/2}}{2} \tag{50}$$

Now we substitute back in our original values for x, a, b and c. We label the resulting resonant frequencies:

$$\omega_\pm^2 = \frac{\omega_p^2 + \omega_h^2}{2} \pm \frac{(\omega_p^4 - 2\omega_p^2\omega_h^2 + \omega_h^4 + 4\omega_{ph}^4)^{1/2}}{2} \tag{51}$$

The numerator in the root can be simplified

$$\omega_\pm^2 = \frac{\omega_p^2 + \omega_h^2}{2} \pm \frac{\left[(\omega_p^2 - \omega_h^2)^2 + 4\omega_{ph}^4\right]^{1/2}}{2} \tag{52}$$

The resonant frequencies of the coupled system can therefore be calculated if the frequencies ω_p, ω_h, and ω_{ph} (or ω_a) are known. By adding the expressions for ω_+^2 and ω_-^2, we obtain an important equation that will be used to calculate the value of ω_p.

$$\omega_+^2 + \omega_-^2 = \omega_p^2 + \omega_h^2 \tag{53}$$

Now we introduce quantities related to damping, γ_+ and γ_-. I will show that these correspond to the first and second resonances that are observed experimentally.

$$\gamma_+ = [(1 + G)/2G][\gamma_p + [(G - 1)/(G + 1)]\gamma_a] \tag{54}$$

$$\gamma_- = [(1+G)/2G][\gamma_a + [(G-1)/(G+1)]\gamma_p] \tag{55}$$

with

$$G = (\omega_+^2 - \omega_-^2)/(\omega_p^2 - \omega_h^2) \tag{56}$$

In Appendix A, I use these quantities show that the denominator D can be written in terms of γ_+ and γ_-. The result is

$$D = (\omega_+^2 - \omega^2 + i\omega\gamma_+)(\omega_-^2 - \omega^2 + i\omega\gamma_-) \tag{57}$$

Now we will calculate and graph the sound pressure level, the top plate mobility and phase versus frequency. To generate these three graphs, measurements of the frequencies of the two lowest resonances and the width of the frequency response curves at those frequencies are required. First, we will calculate and plot the sound pressure level.

Assuming a standard tuning, the wavelengths of the sound waves generated by playing plucking the strings of an acoustic guitar are between 1 and 4 meters. These wavelengths are considerably larger than the lower bout of the guitar, so we can approximate the acoustical radiation generated by a guitar as that generated by a simple source. The sound pressure at a distance R from a simple source radiating into a solid angle of 4π is[4]

$$p = -i\omega\rho U/4\pi R \tag{58}$$

where U is the total volume velocity of the source described by

$$U = AU_p + SU_a \tag{59}$$

Substituting Eqs. (35) and (43) for U_p and U_a into Eq. (59) yields

$$U = A(i\omega)(F/m_p)\left[[(\omega_h^2 - \omega^2) + i\omega\gamma_a]/D\right] + S(-i\omega)(F/m_p)(A/S)(\omega_h^2/D) \tag{60}$$

Canceling appropriate terms, the expression for U becomes

$$U = \frac{Ai\omega F}{Dm_p}(i\omega\gamma_a - \omega^2) \tag{61}$$

Substituting Eq.(61) into Eq.(58) yields

$$p = \frac{-i\omega\rho}{4\pi R}\left(\frac{Ai\omega F}{Dm_p}\right)(i\omega\gamma_a - \omega^2) \tag{62}$$

The i^2 cancels the negative sign and the result is

$$p = \frac{\omega^3 A\rho F}{4\pi R D m_p}(i\gamma_a - \omega) \tag{63}$$

Assuming that γ_a is far smaller than ω,

$$p = -\frac{\omega^4 A\rho F}{4\pi R D m_p} \tag{64}$$

To graph the sound pressure with respect to ω we must determine the magnitude of p. Multiplying the denominator, $D = D_1 + iD_2$, by its complex conjugate we obtain

$$\frac{1}{D_1 + iD_2} \times \frac{D_1 - iD_2}{D_1 - iD_2} = \frac{D_1 - iD_2}{D_1^2 + D_2^2} \tag{65}$$

For the magnitude we square the real and imaginary parts, add them together and take the square root. Fortunately, this leaves us with

$$\left|\frac{1}{D}\right| = \left(\frac{1}{D_1^2 + D_2^2}\right)^{1/2} \tag{66}$$

The magnitude of the pressure is therefore:

$$|p| = \frac{\omega^4 A\rho F}{4\pi R m_p}\left(\frac{1}{D_1^2 + D_2^2}\right)^{1/2} \tag{67}$$

I used Mathematica to generate my plots because of personal familiarity with the program. Because it is a more common experimental practice to control frequency, I have plotted the quantities of interest versus frequency and not angular frequency. Additionally, the commonly measured sound pressure level, in units of decibels (dB), is related to the pressure by

$$SPL = 20\text{Log}_{10}\left(\frac{p}{p_0}\right)\text{dB} \tag{68}$$

where p_0 is a reference pressure of 20 μPa. Inserting our expression for pressure into this equation and graphing SPL as a function of frequency yields the plot in Fig. 2.

Next we calculate the top plate mobility level versus the driving frequency. We will examine the top plate because it provides easily measurable changes. The top plate velocity is

$$U_p = i\omega(F/m_p)\left[[(\omega_h^2 - \omega^2) + i\omega\gamma_a]/D\right] \tag{69}$$

We need a value for γ_a, so we revisit Eq. (54):

$$\gamma_+ = [(1+G)/2G]\,[\gamma_p + [(G-1)/(G+1)]\gamma_a] \tag{70}$$

First we solve for γ_p in terms of γ_a and γ_+.

$$\gamma_p = \frac{2G\gamma_+}{(1+G)} - \frac{(G-1)}{(G+1)}\gamma_a \tag{71}$$

Now we insert this expression into Eq. (55)

$$\gamma_- = \left[\frac{(1+G)}{2G}\right]\left\{\gamma_a + \frac{(G-1)}{(G+1)} \times \left[\frac{2G\gamma_+}{(1+G)} - \frac{(G-1)}{(G+1)}\gamma_a\right]\right\} \tag{72}$$

Distributing $(1+G)$ and multiplying through by 2G we have

$$2G\gamma_- = (1+G)\gamma_a + (G-1) \times \left[\frac{2G\gamma_+}{(1+G)} - \frac{(G-1)}{(G+1)}\gamma_a\right] \tag{73}$$

Distributing γ_a in the last term on the right and multiplying both sides by $(G+1)$ yields

$$2G\gamma_-(G+1) = (G+1)^2\gamma_a + (G-1)[2G\gamma_+ - \gamma_a G + \gamma_a] \tag{74}$$

Dividing both sides by $(G-1)$ and moving the γ_a term to the left side

$$\gamma_a\left[\frac{(G+1)^2}{(G-1)} - G + 1\right] = \frac{2G\gamma_-(G+1)}{(G-1)} - 2G\gamma_+ \tag{75}$$

Now it should be evident that solving for γ_a requires only simple algebraic manipulation, the result of which is

$$\gamma_a = \frac{1}{2}\left[(G+1)\gamma_- + (G-1)\gamma_+\right] \tag{76}$$

And upon insertion of our value for G we arrive at

$$\gamma_a = \frac{1}{2}\left(\frac{\omega_+^2 - \omega_-^2}{\omega_p^2 - \omega_h^2} + 1\right)\gamma_- + \frac{1}{2}\left(\frac{\omega_+^2 - \omega_-^2}{\omega_p^2 - \omega_h^2} - 1\right)\gamma_+ \tag{77}$$

Now that we have an equation to describe γ_a we return to our equation for the top plate velocity. The final task before plotting is determining the magnitude of U_p. Returning to Eq. (35)

$$U_p = i\omega(F/m_p)\left[[(\omega_h^2 - \omega^2) + i\omega\gamma_a]/D\right] \tag{78}$$

Distributing the first i will produce identical results so it can be ignored in the subsequent calculations. The magnitude of U_p is the square root of the sum of the squares of both real and imaginary terms.

$$|U_p| = \left(\frac{\omega F}{m_p}\right) \frac{\left[(\omega_h^2 - \omega^2)^2 + (\omega^2\gamma_a^2)^2\right]^{1/2}}{|D|} \tag{79}$$

Inserting our expression from Eq. (66) we arrive at

$$|U_p| = \left(\frac{\omega F}{m_p}\right) \left[\frac{(\omega_h^4 - 2\omega_h^2\omega^2 + \omega^4 + \omega^2\gamma_a)^{1/2}}{(D_1^2 + D_2^2)^{1/2}}\right] \tag{80}$$

Defining the top plate mobility, M, as the ratio of the top plate velocity and driving force,

$$M = \frac{u_p}{f} = \frac{U_p}{F} \tag{81}$$

we can now plot the top plate mobility level as a function of the driving frequency in Fig. 3. To make the correlation between the sound pressure level plot and the top plate mobility level plot evident, I've multiplied the base ten logarithm of top plate mobility level by 20 as I've done with SPL.

Next we will calculate the phase Φ of the top plate velocity relative to the driving force. The phase of a complex expression can be calculated by taking the inverse tangent of the imaginary component divided by the real component. Mathematically

$$\Phi(u_p/f\,) = \tan^{-1}\left[\frac{\mathrm{Im}(u_\mathrm{p}/f)}{\mathrm{Re}(u_\mathrm{p}/f)}\right] \tag{82}$$

Recall that

$$U_p = \frac{i\omega F}{m_p} \times \left[(\omega_h^2 - \omega^2) + i\omega\gamma_a\right] \times \frac{D_1 - iD_2}{D_1^2 + D_2^2} \tag{83}$$

Distributing the numerator on the right and i on the left and dividing through by F we have

$$U_p/F \;= \frac{\omega}{m_p(D_1^2 + D_2^2)} \times \left[(D_1\omega\gamma_a - D_2\omega_h^2 + D_2\omega^2) + i(D_1\omega_h^2 - D_1\omega^2 + \omega D_2\gamma_a)\right] \tag{84}$$

The ratio of the imaginary component to the real component conveniently cancels out the prefactor, leaving

$$\Phi(U_p/F\,) = \tan^{-1}\left[\frac{D_1\omega_\mathrm{h}^2 - D_1\omega^2 + \omega D_2\gamma_\mathrm{a}}{D_1\omega\gamma_\mathrm{a} - D_2\omega_\mathrm{h}^2 + D_2\omega^2}\right] \tag{85}$$

From this equation we plot the phase of the top plate velocity versus the frequency in Fig. 4.

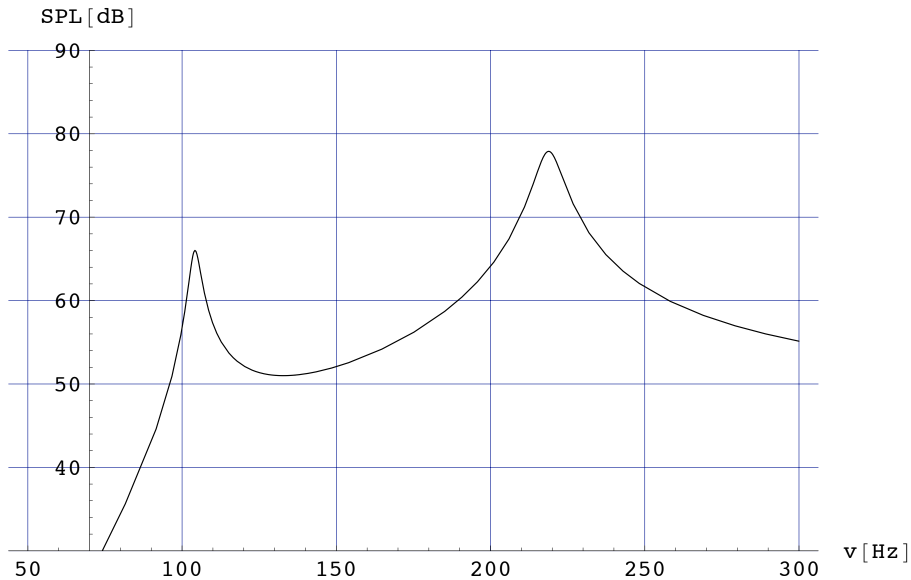

FIG. 2: Sound pressure level versus driving frequency. Here, $f_h = 127$ Hz, $f_- = 104$ Hz, $f_+ = 219$ Hz, $m_\mathrm{p} = 112$ g, $\gamma_- = 22.5$ rad/s, $\gamma_+ = 53.3$ rad/s.

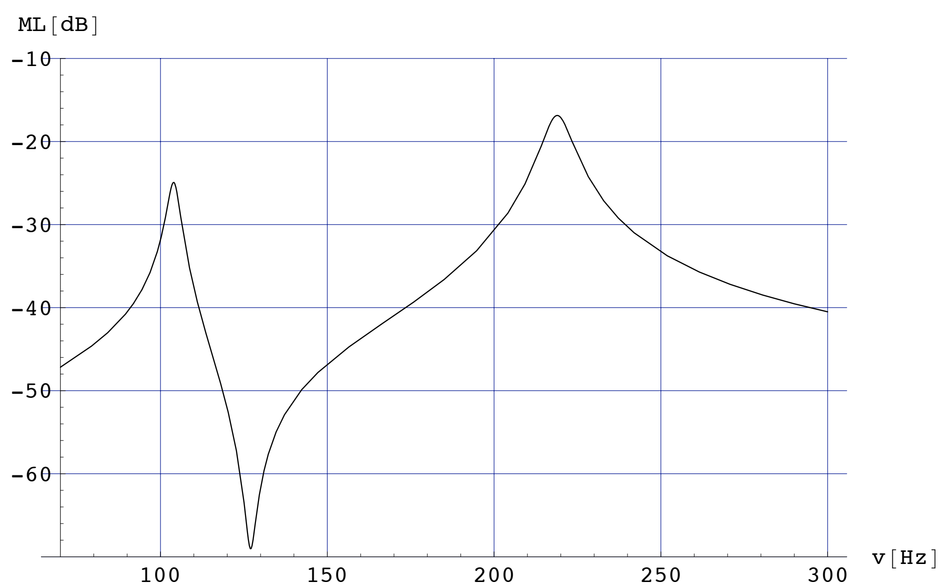

FIG. 3: Mobility level of the top plate versus driving frequency. The same values that were used in Fig. 2 were used to calculate the mobility level.

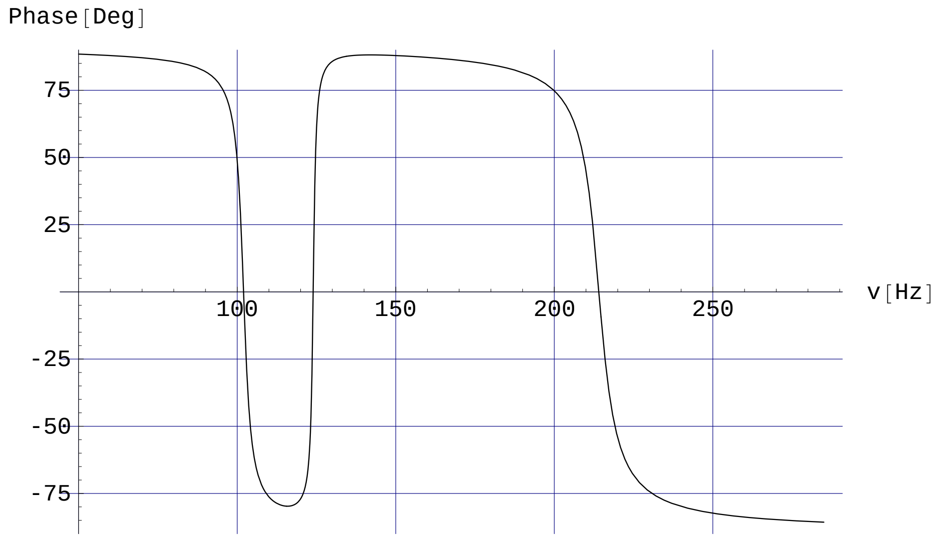

FIG. 4: Phase of the top plate velocity relative to the driving force versus driving frequency. A few different parametric values were used to calculate the phase. Here, $f_h = 124$ Hz, $f_- = 102$ Hz, $f_+ = 214$ Hz, $m_\mathrm{p} = 112$ g, $\gamma_- = 22.01$ rad/s, $\gamma_+ = 52.12$ rad/s. Note that the phase becomes negative at both f_- and f_+, shortly after 100 Hz and 200 Hz respectively.

Plotting these equations required numerical values for the following variables: ω_h, ω_-, ω_+, m_p, γ_- and γ_+. Because frequency is more commonly measured than angular frequency, our experimental values for ω_h, ω_- and ω_+ are defined in terms of f_h, f_- and f_+.

The values for f_h, the Helmholtz resonance frequency of the air cavity of the guitar, can be calculated given the dimensions of the body of the guitar. It is more easily and accurately obtained, along with the first two resonance frequencies of the guitar, f_- and f_+, by driving the top plate and observing the amplitude of top plate displacement and velocity. The mass of the top plate, m_p, can be measured directly or calculated using f_p assuming the stiffness of the top plate is known.

The values for γ_- and γ_+ are obtained similarly. The value for γ_- is calculated using

$$\gamma_- = 2\pi\delta f_- \tag{86}$$

and measuring the frequency difference δf_- between the 3-dB limits of the first resonance, f_-, in the sound pressure level. Similar measurement and calculation yields the value for γ_+.

In the absence of equipment for which these measurements could be made I have used values obtained through experiment by Christensen and Vistisen.

III. RESULTS AND DISCUSSION

The data of Christensen and Vistisen provided the values used to generate the three plots. Our calculations therefore yielded identical graphs for sound pressure level, mobility level and phase, verifying the accuracy of our calculations.

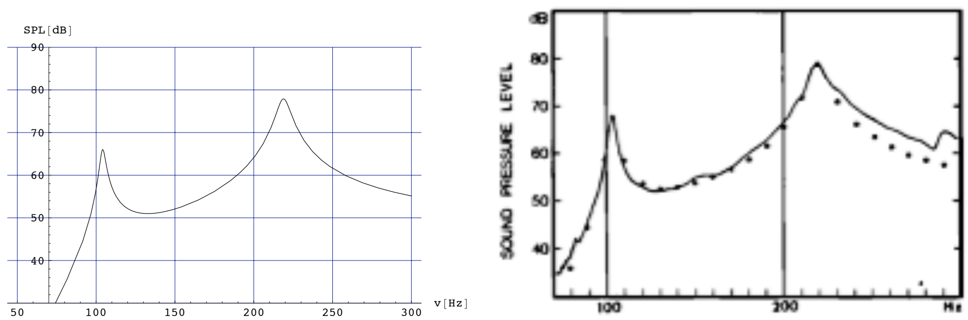

FIG. 5: The plot on the left is the sound pressure level versus driving frequency plot calculated in this paper. On the right is the same plot generated by Christensen and Vistisen; they represent their theoretical results with dots and their experimental data with a solid line.

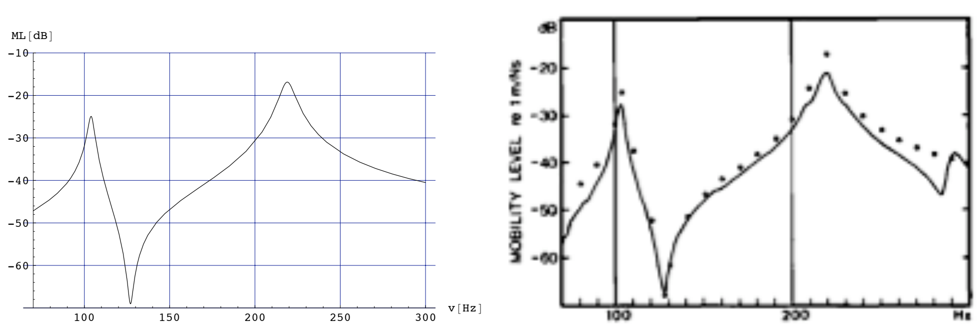

FIG. 6: The plot on the left is the mobility level versus the driving frequency plot calculated in this paper. The plot on the right is the same plot generated by Christensen and Vistisen; they represent their theoretical results with a dotted line and experimental data with a solid line.

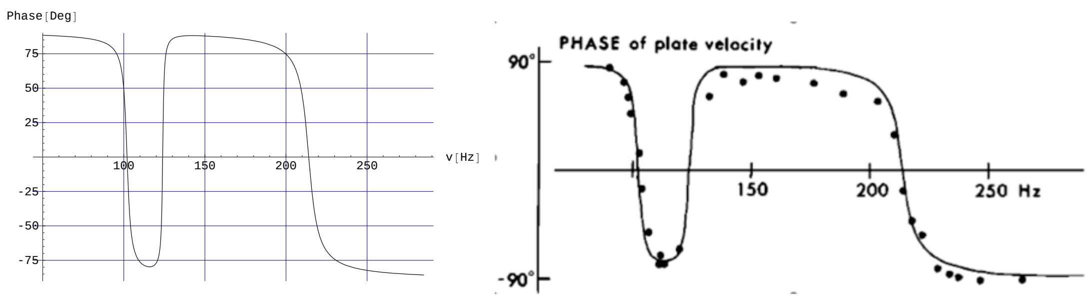

FIG. 7: The plot on the left is the phase of the mobility versus the driving frequency as calculated in this paper. The plot on the right is the same plot generated by Christensen and Vistisen; they represent their theoretical results with a solid line and their experimental data with dots.

Unexplained by our model are the tertiary resonances evident in the experimental data of Christensen and Vistisen. These tertiary resonances are observable in the sound pressure level and mobility level plots (Fig. 5 and Fig. 6). I suspect that a similar theoretical model for the acoustic guitar incorporating a third coupled oscillator, namely the back plate, would display a tertiary resonance.

The primary weakness of this model is its utility, which appears to be limited to academic study. The reason for this weakness is that the values for f_+, f_-,γ_+ and γ_- can only be obtained experimentally; a measurement of the frequency response of the guitar must be made to obtain these values. A theoretical model reliant on experimental values would be of little practical value, especially to someone seeking to improve the method of guitar construction. For example, a luthier - seeking to alter the sound of a guitar by varying the material of the top plate - would be interested in the predicted frequency response of the guitar prior to construction. Because of the primary weakness of this model, the luthier would have to first construct the guitar and then make measurements to obtain the values for f_-, f_+, γ_- and γ_+. Once the experimental values are obtained, a few calculations reveal one use for the model. The values for f_+, f_-,γ_+ and γ_- can be used in conjunction with the calculated values w_p and w_h to determine R_p and R_a, values characteristic of the material of the top plate and air column respectively. Damping coefficients of the top plate and air piston would first be calculated. Multiplying the damping coefficients by their respective masses, obtained through density and dimension measurements, will yield their respective resistance to motion.

APPENDIX A: INTRODUCING TWO NEW GAMMAS

In this section I show that the implementation of new variables γ_+ and γ_- into our model is mathematically sound; I do this by equating the expression for the denominator with a proposed new expression which includes the new variabes. By comparing the coefficients of all variables I demonstrate the validity of the substitution of our new expression.

We begin with our old expression for the denominator.

$$D = (\omega_p^2 - \omega^2 + i\omega\gamma_p)(\omega_h^2 - \omega^2 + i\omega\gamma_a) - \omega_{ph}^4 \tag{A1}$$

Distributing, we have

$$D = \omega_p^2\omega_h^2 - \omega^2\omega_p^2 + i\omega\omega_p^2\gamma_a - \omega^2\omega_h^2 + \omega^4 - i\omega^3\gamma_a + i\omega\omega_h^2\gamma_p - i\omega^3\gamma_p - \omega^2\gamma_p\gamma_a - \omega_{ph}^4 \tag{A2}$$

Grouping real and imaginary terms we arrive at

$$D = (\omega_p^2\omega_h^2 - \omega^2\omega_p^2 - \omega^2\omega_h^2 + \omega^4 - \omega^2\gamma_p\gamma_a - \omega_{ph}^4) + i(\omega\omega_p^2\gamma_a - \omega^3\gamma_p - \omega^3\gamma_a + \omega\omega_h^2\gamma_p) \tag{A3}$$

Next we examine our new expression for the denominator that includes γ_+ and γ_-.

$$D = (\omega_+^2 - \omega^2 + i\omega\gamma_+)(\omega_-^2 - \omega^2 + i\omega\gamma_-) \tag{A4}$$

Distributing, we have

$$D = \omega_+^2\omega_-^2 - \omega^2\omega_+^2 + i\omega\omega_+^2\gamma_- - \omega^2\omega_-^2 + \omega^4 - i\omega^3\gamma_- + i\omega\omega_-^2\gamma_+ - i\omega^3\gamma_+ - \omega^2\gamma_+\gamma_- \tag{A5}$$

Grouping real and imaginary terms

$$D = (\omega_+^2\omega_-^2 - \omega^2\omega_+^2 - \omega^2\omega_-^2 + \omega^4 - \omega^2\gamma_+\gamma_-) + i(\omega\omega_+^2\gamma_- - \omega^3\gamma_- + \omega\omega_-^2\gamma_+ - \omega^3\gamma_+) \tag{A6}$$

We equate the imaginary parts from Eqs. (A3) and (A6).

$$\omega\omega_p^2\gamma_a - \omega^3\gamma_p - \omega^3\gamma_a + \omega\omega_h^2\gamma_p = \omega\omega_+^2\gamma_- - \omega^3\gamma_- + \omega\omega_-^2\gamma_+ - \omega^3\gamma_+ \tag{A7}$$

A factor of ω cancels from each term. Again looking at just the ω^2 terms that remain

$$\omega^2\gamma_p + \omega^2\gamma_a = \omega^2\gamma_+ + \omega^2\gamma_- \tag{A8}$$

which implies that

$$\gamma_p + \gamma_a = \gamma_+ + \gamma_- \tag{A9}$$

Another important result of equating imaginary terms is what remains when the ω^2 terms are removed.

$$\omega_p^2\gamma_a + \omega_h^2\gamma_p = \omega_+^2\gamma_- + \omega_-^2\gamma_+ \tag{A10}$$

Solving Eq. (A9) for γ_-,

$$\gamma_- = \gamma_a + \gamma_p - \gamma_+ \tag{A11}$$

Combining this result with Eq. (A10), we obtain

$$\omega_p^2\gamma_a + \omega_h^2\gamma_p = \omega_+^2(\gamma_a + \gamma_p - \gamma_+) + \omega_-^2\gamma_+ \tag{A12}$$

Solving this equation for γ_+,

$$\gamma_+ = \frac{\gamma_a(\omega_p^2 - \omega_+^2) + \gamma_p(\omega_h^2 - \omega_+^2)}{\omega_-^2 - \omega_+^2} \tag{A13}$$

Leaving this equation for a moment, we now rearrange Eq. (54). First we multiply through by 2G and distribute (1+G).

$$2G\gamma_+ = (1+G)\gamma_p + (G-1)\gamma_a \tag{A14}$$

Next we substitute the definition of G from Eq. (56)

$$2\left(\frac{\omega_+^2 - \omega_-^2}{\omega_p^2 - \omega_h^2}\right)\gamma_+ = \left(1 + \frac{\omega_+^2 - \omega_-^2}{\omega_p^2 - \omega_h^2}\right)\gamma_p + \left(\frac{\omega_+^2 - \omega_-^2}{\omega_p^2 - \omega_h^2} - 1\right)\gamma_a \tag{A15}$$

Substituting our expression for γ_+ from Eq.(A13) into this equation we obtain

$$2\left(\frac{\omega_+^2 - \omega_-^2}{\omega_p^2 - \omega_h^2}\right)\left[\frac{\gamma_a(\omega_p^2 - \omega_+^2) + \gamma_p(\omega_h^2 - \omega_+^2)}{\omega_-^2 - \omega_+^2}\right] = \left(1 + \frac{\omega_+^2 - \omega_-^2}{\omega_p^2 - \omega_h^2}\right)\gamma_p + \left(\frac{\omega_+^2 - \omega_-^2}{\omega_p^2 - \omega_h^2} - 1\right)\gamma_a \tag{A16}$$

Canceling terms of $(\omega_+^2 - \omega_-^2)$ and multiplying through by $(\omega_p^2 - \omega_h^2)$ yields

$$-2\left[\gamma_a(\omega_p^2 - \omega_+^2) + \gamma_p(\omega_h^2 - \omega_+^2)\right] = (\omega_p^2 - \omega_h^2 + \omega_+^2 - \omega_-^2)\gamma_p + (\omega_+^2 - \omega_-^2 - \omega_p^2 + \omega_h^2)\gamma_a \tag{A17}$$

Distributing the two, multiplying both sides by -1 and rearranging a few terms on the right hand side we obtain

$$2\gamma_a(\omega_p^2 - \omega_+^2) + 2\gamma_p(\omega_h^2 - \omega_+^2) = \gamma_a(\omega_p^2 - \omega_+^2 + \omega_-^2 - \omega_h^2) + \gamma_p(\omega_h^2 - \omega_+^2 + \omega_-^2 - \omega_p^2) \tag{A18}$$

Recall from Eq. (53) that

$$\omega_+^2 + \omega_-^2 = \omega_p^2 + \omega_h^2 \tag{A19}$$

Rearranging this equation yields

$$\omega_-^2 - \omega_h^2 = \omega_p^2 - \omega_+^2 \tag{A20}$$

will replace the latter two terms in the γ_a expression on the right. Also,

$$\omega_-^2 - \omega_p^2 = \omega_h^2 - \omega_+^2 \tag{A21}$$

which we'll use to replace the latter two terms in the γ_p expression on the right. Doing so we obtain

$$2\gamma_a(\omega_p^2 - \omega_+^2) + 2\gamma_p(\omega_h^2 - \omega_+^2) = \gamma_a(\omega_p^2 - \omega_+^2 + \omega_p^2 - \omega_+^2) + \gamma_p(\omega_h^2 - \omega_+^2 + \omega_h^2 - \omega_+^2) \tag{A22}$$

Each expression yields a factor of 2 such that

$$2\gamma_a(\omega_p^2 - \omega_+^2) + 2\gamma_p(\omega_h^2 - \omega_+^2) = 2\gamma_a(\omega_p^2 - \omega_+^2) + 2\gamma_p(\omega_h^2 - \omega_+^2) \tag{A23}$$

Having showed that both sides are equivalent, we are justified in making the γ_+ and γ_- substitutions.

APPENDIX B: MATHEMATICA CODE

I used Mathematica to generate the plots for sound pressure level, mobility level and phase. Below is the code pertaining to each plot. Note that the following substitutions were made in the Mathematica code: $a = f_+$, $b = f_-$, $c = \gamma_+$, $d = \gamma_-$, $e = f_h$ and $x = f$.

```
Plot[a = 219 * 2 Pi; b = 104 * 2 Pi; c = 53.3340148168; d = 22.5328024809;
 r = (a^2) * (b^2) - (a^2 + b^2 - ((2 Pi * x) ^2) - c * d) * ((2 Pi * x) ^2);
 i = (2 Pi * x) * ((a^2) * d - ((2 Pi * x) ^2) * d + (b^2) * c - ((2 Pi * x) ^2) * c);
 20 * Log[10, ((0.004708925827) * ((2 Pi * x) ^4) / ((r^2 + i^2) ^ (1 / 2))) / (20 * (10^-6))],
 {x, 50, 300}, PlotRange → {30, 90}, GridLines → Automatic,
 ImageSize → {500, Automatic}, AxesLabel → {v[Hz], SPL[dB]}, AxesOrigin → {70, 30}]
```

FIG. 8: The Mathematica code for generating the sound pressure level vs. frequency plot.

```
Plot[a = 219 * 2 Pi;  b = 104 * 2 Pi; c = a / 25.8; d = b / 29.0; w = 2 Pi * x;
 dr = (a^2) * (b^2) - (w^2) (a^2 + b^2) + w^4; di = w ((a^2) * d + (b^2) * c - (w^2) (c + d));
 e = 127 * 2 Pi; f = 0.5 * (- (j - 1.0) * c + (j + 1.0) * d); j = (a^2 - b^2) / (p^2 - e^2);
 p = (a^2 + b^2 - e^2) ^ (1 / 2); nr = ((e^2 - w^2) * dr + (2 Pi * x * f * di));
 ni = ((w * f * dr) - (e^2 - w^2) * di);
 20 * Log[10, (w / .112) * ((nr^2 + ni^2) ^ (1 / 2)) / (dr^2 + di^2)],
 {x, 70, 300}, PlotRange → {-70, -10}, GridLines → Automatic,
 ImageSize → {500, Automatic}, AxesLabel → {v [Hz], ML [dB]}, AxesOrigin → {70, -70}]
```

FIG. 9: The Mathematica code for generating the mobility level vs. driving frequency plot.

```
Plot[a = 214 * 2 Pi; b = 102 * 2 Pi; c = 52.12; d = 22.099;
 e = 124 * 2 Pi; f = 7.9137; g = (a^2) * (b^2) - (a^2) * ((2 Pi * x) ^2) -
   (b^2) * ((2 Pi * x) ^2) + ((2 Pi * x) ^4) - ((2 Pi * x) ^2) * (c * d);
 h = (2 Pi * x) * (a^2) * (d) - ((2 Pi * x) ^3) * (d) + (2 Pi * x) * (b^2) * (c) - ((2 Pi * x) ^3) * (c);
 ArcTan[- (g * (e^2) - g * ((2 Pi * x) ^2) + 2 Pi * x * h * f) /
    (g * (2 Pi * x) * f - h * (e^2) + h * ((2 Pi * x) ^2))] / °, {x, 50, 285},
 PlotRange → {-90, 90}, GridLines → Automatic, ImageSize → {500, Automatic},
 AxesLabel → {v[Hz], Phase [Deg]}, AxesOrigin → {50, 0}]
```

FIG. 10: The Mathematica code for generating the phase vs. driving frequency plot.

APPENDIX C: GRADUATE RESEARCH FAIR PRESENTATION

On March 26, 2007, I presented my calculations at the Eastern Michigan University Graduate Research Fair. Fig. 11 is the poster I presented at the fair. Notice on the poster that at the time of the presentation I had produced graphs that resembled that of Christensen and Vistisen but was unable to match their units. Additionally, the limitations of the model were yet to be realized and a three oscillator model was intended to be attempted, incorporating the back plate of the guitar. Since the presentation, I discovered unsuccessful attempts by others to use the discrete model described in this paper to permit a third degree of freedom. Finite element analysis, however, has been successfully implemented to produce a three oscillator model that matches experimental results.[3]

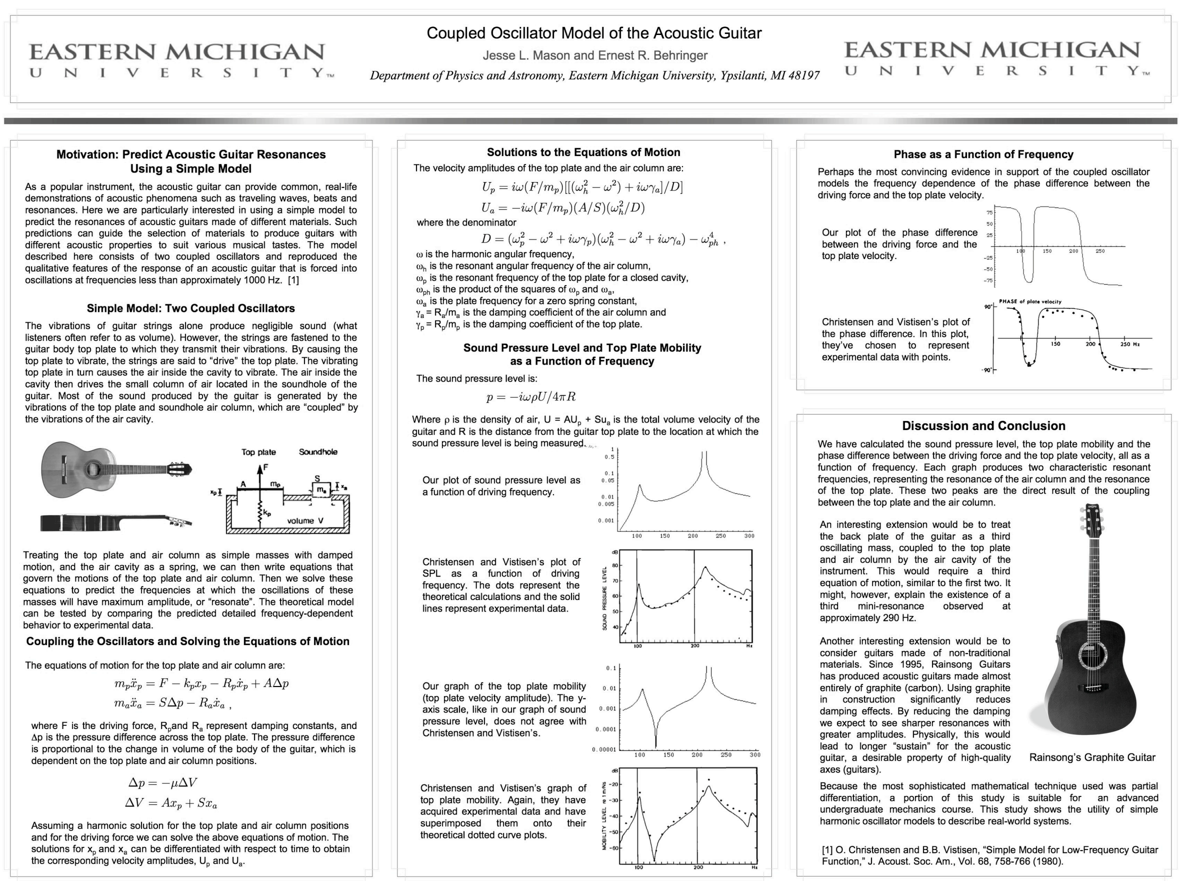

FIG. 11: The poster I created for the 2007 Graduate Research Fair. Note the dissimilarities between the Christensen and Vistisen model and the model that I calculated.

ACKNOWLEDGMENTS

I would like to extend resonating thanks to Dr. Ernest Behringer, whose tireless efforts unveiled a model of scientific excellence and without whom this paper would not have been possible.

[1] Ove Christensen and Bo Vistisen. Simple model for low-frequency guitar function. *J. Acoust. Soc. Am.*, **68**(3):758–765, 1980.

[2] Neville H. Fletcher and Thomas D. Rossing. *The Physics of Musical Instruments.* Springer Publishing Company, 1998.

[3] Mark French. Structural Modification of Stringed Instruments. *Mechanical Systems and Signal Processing*, **21**(1):98–107, 2007.

[4] Daniel Russell. Measuring the Directivity Patterns of a Loudspeaker with and without a Baffle. http://www.gmi.edu/ drussell/GMI-Acoustics/Directivity2.html, December 2007.